Vibration Analysis of Tapered Beam

A Thesis Submitted in Partial Fulfilment

of the Requirements for the Award of the Degree of

Master of Technology

in

Machine Design and Analysis

by

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Rourkela-769008, Odisha, INDIA
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CERTIFICATE

This is to certify that the thesis entitled "**VIBRATION ANALYSIS OF TAPERED BEAM**" by **Rishi Kumar Shukla**, submitted to the National Institute of Technology (NIT), Rourkela for the award of Master of Technology in **Machine Design and Analysis**, is a record of bona fide research work carried out by him in the Department of Mechanical Engineering, under our supervision and guidance.

I believe that this thesis fulfills part of the requirements for the award of degree of Master of Technology. The results embodied in the thesis have not been submitted for the award of any other degree elsewhere.

Place: Rourkela

Date:

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ABSTRACT

Beams are very common types of structural components and it can be classified according to their geometric configuration as uniform or taper and slender or thick. If practically analyzed, the non-uniform beams provide a better distribution of mass and strength than uniform beams and can meet special functional requirements in architecture, aeronautics, robotics, and other innovative engineering applications. Design of such structures is important to resist dynamic forces, such as wind and earthquakes. It requires the basic knowledge of natural frequencies and mode shapes of those structures. In this research work, the equation of motion of a double tapered cantilever Euler beam is derived to find out the natural frequencies of the structure. Finite element formulation has been done by using Weighted residual and Galerkin's method. Natural frequencies and mode shapes are obtained for different taper ratios. The effect of taper ratio on natural frequencies and mode shapes are evaluated and compared.

Keywords: Tapered beam, FEM, Taper ratio, Galerkin's Method, Mode shapes.

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LIST OF NOMENCLATURE

Symbols	Description
I_O	Moment of inertia at the fixed end
I_1	Moment of inertia at the free end
b_0	Width of the beam at the fixed end
b_1	Width of the beam at the free end
h_0	Depth of the beam at the fixed end
h_1	Depth of the beam at the free end
A_x	Cross-sectional area at length x from free end
r	Dimensionless number(h_0/ h_1)-1
$[K^e]$	Element stiffness matrix
$[M^{e]}$	Consistent mass matrices
$H_i(x)$	Hermitian shape function
d^e	Nodal degree of freedom vector
ω	Natural frequency
α	h_0/ h_1
β	b_0/ b_1
$\{ \phi \}$	Mode shape vector
Ψ^e	Element domain
n	The number of elements for the beam

INTRODUCTION

1.1 Introduction

It is well known that beams are very common types of structural components and can be classified according to their geometric configuration as uniform or tapered, and slender or thick. It has been used in many engineering applications and a large number of studies can be found in literature about transverse vibration of uniform isotropic beams. But if practically analyzed, the non-uniform beams may provide a better or more suitable distribution of mass and strength than uniform beams and therefore can meet special functional requirements in architecture, aeronautics, robotics, and other innovative engineering applications and they has been the subject of numerous studies. Non-prismatic members are increasingly being used in diversities as for their economic, aesthetic, and other considerations.

Design of such structures to resist dynamic forces, such as wind and earthquakes, requires a knowledge of their natural frequencies and the mode shapes of vibration. The vibration of tapered beam linearly in either the horizontal or the vertical plane finds wide application for electrical contacts and for springs in electromechanical devices.

For the tapered beam vibration analysis Euler beam theory is used. Free vibration analysis that has been done in here is a process of describing a structure in terms of its natural characteristics which are the frequency and mode shapes. The change of modal characteristics directly provides an indication of structural condition based on changes in mode shapes and frequencies of vibration.

There have been many methods developed yet now for calculating the frequencies and mode shapes of beam. Due to advancement in computational techniques and availability of software, FEA is quite a less cumbersome than the conventional methods. Prior to development of the Finite Element Method, there existed an approximation technique for solving differential equations called the Method of Weighted Residuals (MWR). This method is presented as an introduction, before using a particular subclass of MWR, the Galerkin's Method of Weighted Residuals, to derive the element equations for the finite element method. These formulations have displacements and rotations as the primary nodal variables, to satisfy the continuity requirement each node has both deflection and slope as nodal variables. Since there are four nodal variables for the beam element, a cubic polynomial function is assumed. The clamped free beam that is being considered here is assumed to be homogeneous and isotropic. Beam of rectangular cross-section area with width and depth varying linearly are taken and elemental mass and stiffness matrices are being derived. The effects of taper ratio on the fundamental frequency and mode shapes are shown in with comparison for clamped-free and simply supported beam via graphs and tables. These results are then compared with the available analytical solutions.

LITERATURE REVIEW

2.1 Literature review

Mabie and Rogers [1] developed the differential equation from the Bernoulli-Euler equation for the free vibration of a tapered cantilever beam. The beam tapers linearly in the horizontal and in the vertical planes simultaneously. The effects of different taper ratio on the vibration frequency have been analyzed.

Mabie and Rogers [2] studied the free vibrations of non-uniform cantilever beams with an end support have been investigated, using the equations of Bernoulli-Euler. Two configurations of interest are treated in their analysis (a) constant width and linearly variable thickness and (b) constant thickness and linearly variable width. Charts have been plotted for each case.

Sharp and Cobble [3] derived the equation of motion of beam for a uniform-cross-section damped beam elastically restrained against rotation at either ends. This has been solved for the displacement under very general arbitrary initial and distributed load conditions. The solution is based on the properties of Hermitian operator.

Carnegie &Thomas [4] determined the natural frequencies and mode shapes of vibration of pre-twisted, tapered cantilever blades which is of great importance in the design of many engineering components. These include turbine blading, compressor blading, aircraft propeller blades, and helicopter rotor blades.

Chun [5] considered the free vibration of a beam hinged at one end by a rotational spring with a constant spring constant and the other end free. The beam includes the ‘simply supported-free’ beam and the ‘clamped-free’ beam as the limiting cases of the zero spring constant and the infinite spring constant, respectively. Normal functions derived here can be of use to an approximate analysis for rectangular plate vibrations when one pair of the parallel edges is of the spring hinged-free type.

Goel [6] investigated the transverse vibrations of linearly tapered beams, elastically restrained against rotation at either end. He studied the vibration characteristics of beam which carries a concentrated mass. He assumed that one end of the beam is free and the other end is hinged by a rotational spring of constant stiffness. Results for the first three eigen frequencies with different values of stiffness ratios (ratio of spring stiffness and beam stiffness at either end) and taper ratio are presented. Cases of a tapered cantilever beam with a concentrated mass at the free end and spring hinged at the other end have also been presented.

Hibbeler [8] studied the free vibration analysis of a beam having a combination of clamped or ideally pinned end supports. In many real cases, however, beams are subjected to a certain amount of bending stiffness at their end support. He analyzed and considered such a case assuming the beam to be spring-hinged at both ends. The general frequency equation and the normal mode function are derived for the case when the spring stiffness at each support is different. The first five roots of this equation are computed and presented in a tabulated form so that the roots may be obtained for a variety of spring support and boundary conditions.

Gupta [9] developed the stiffness and consistent mass matrices for linearly tapered beam element of any cross-sectional shape. Variation of area and moment of inertia of the cross section along the axis of the element was exactly represented by simple functions involving shape factors.

Raju et al. [10] studied the free vibration analysis of beam using the simple finite element formulation. They applied it to the large amplitude vibrations for different conditions i.e. simply supported and clamped tapered beams with linearly varying breadth and depth tapers.

Gupta & Rao [11] derived the stiffness and mass matrices of a twisted beam element with linearly varying breadth and depth. The angle of twist was assumed to vary linearly along the length of the beam. The effects of shear deformation and (rotary inertia has been considered in deriving the elemental matrices. The first four natural frequencies and mode shapes have been calculated for cantilever beams of various depth and breadth taper ratios at different angles of twist. The results were compared with those available.

Bailey [12] presented the analytical solution for the vibration of non-uniform beams with and without discontinuities and incorporating various boundary conditions. Results obtained were compared with the existing results for certain cases. It has been shown that the direct solution converges to the exact solution.

C. W. S. To [13] derived the explicit expressions for mass and stiffness matrices of two higher order tapered beam elements for vibration analysis. One possesses three degrees of freedom per node and the other four degrees of freedom per node. Thus, this element adequately represents all the physical situations involved in any combination of displacement, rotation, bending moment and shearing force. The eigenvalues obtained by employing the higher order elements converge more rapidly to the exact solution than those obtained by using the lower order one.

Lau [14] calculated the first five natural frequencies and tabulated for a non-uniform cantilever beam with a mass at the free end based on Euler theory. The reliability of the results has been verified by checking the clamped-free case, and the clamped-clamped case and the cantilever beam with end mass case with the results presented previously. The results reported herein may be useful for several engineering situations: e.g., a mast antenna structure, and a tower-tank type structure.

Banerjee and Williams [15] used the Euler-Bernoulli theory and Bessel functions to obtain explicit expressions for the exact static stiffness for axial, torsional and flexural deformation of an axially loaded tapered beam. Procedure were given for calculating the number of critical buckling loads of a clamped-clamped member that are exceeded by any trial load so that an existing algorithm can be used to obtain the exact critical buckling loads of structures.

Liew et al. [18] presented a thorough computational investigation into the effects of initial twist and thickness variation on the vibratory characteristics of cantilevered pre-twisted thin shallow conical shells with generally varying thickness. An energy approach with the Ritz minimization procedure was employed to arrive at a governing eigen value equation. The admissible shape functions which comprise sets of mathematically complete two-dimensional orthogonal polynomials and a basic function are introduced to account for the boundary constraints and to approximate the three-dimensional displacements of the conical shell.

Chaudhari and Maiti [19] did the modeling of transverse vibration of a beam of linearly variable depth and constant thickness in the presence of an `open' edge crack normal to its axis that has been proposed using the concept of a rotational spring to represent the crack section and the Frobenius method to enable possible detection of location of the crack based on the measurement of natural frequencies. A number of numerical examples were presented involving cantilever beams to show the effectiveness of the method for the inverse problem. An analytical solution for the study of vibration of taper beams with crack normal to its axis has been presented. The Frobenius method has been combined with the modeling based on rotational spring for a crack in a linearly variable depth beam having constant thickness. This method can help to solve both the forward and inverse problems. While solving the inverse problem, the method predicts the location of a crack quite accurately. The maximum error in the prediction of the location considering all the cases studied was about 3%. The scheme can be employed to locate an unknown crack in a taper beam. The method can also be used to solve the forward problem.

Ece et al. [21] investigated vibration analysis of an isotropic beam which has a variable cross-section. Governing equation was reduced to an ordinary differential equation in spatial coordinate for a family of cross-section geometries with exponentially varying width. Analytical solutions of the vibration of the beam are obtained for three different types of boundary conditions associated with simply supported, clamped and free ends. Natural frequencies and mode shapes are determined for each set of boundary conditions. It shows that the non-uniformity in the cross-section influences the natural frequencies and the mode shapes.

Gunda et al. [22] developed the numerically efficient super element which was proposed as a low degree of freedom model for dynamic analysis of rotating tapered beams. The element used, was a combination of polynomials and trigonometric functions as shape functions in what is also called the Fourier-p approach. The super element also allows an easy incorporation of polynomial variations of mass and stiffness properties typically used to model helicopter and wind turbine blades.

Shooshtari and Khajavi [24] presented new procedure to find the exact shape functions and stiffness matrices of non-prismatic beam elements for the Euler-Bernoulli and Timoshenko beam. Strain interpolating functions involve low-order polynomials and can suitably track the variations along the beam element. The proposed procedure was implemented to model non-prismatic Euler-Bernoulli and Timoshenko beam elements and is verified by different numerical examples.

Chelirem and Lakhdar [25] studied the free vibration analysis of a rotating double tapered blade that undergoes bending vibration. Variational approach of Lagrange was applied to the derived energy expressions to obtain the governing differential equations of motion and the boundary conditions. Rotational speed, taper ratios were incorporated into the equations of motion. They used finite element method applied to a beam to solve the governing differential equations of motion. The effects of the incorporated parameters on the natural frequencies and forced vibration were investigated.

Attarnejad and Ahmad [26] introduced the concept of basic displacement functions (BDFs) for free vibration analysis of rotating tapered beams. Holding pure structural/mechanical interpretations, BDFs are obtained by solving the governing static differential equation of motion of rotating Euler– Bernoulli beams and imposing appropriate boundary conditions. The method was employed to determine the natural frequencies of tapered rotating beams with different variations of cross-sectional dimensions.

Gavin [28] described the formulation of stiffness and mass matrices for structural elements such as truss bars, beams and plates. The formulation of each element involves the determination of gradients of potential and kinetic energy functions with respect to a set of coordinates defining the displacements at the ends or nodes of the elements. The potential and kinetic energy of the functions were therefore written in terms of these nodal displacements (i.e., generalized coordinates). The distribution of strains and velocities within the element must be written in terms of nodal coordinates as well.

2.2 Objective

Based on the literature review and the scope outlined in the previous sub-section, the following objectives are framed for the present work

- Formulation of governing differential equation of motion of linearly tapered Euler-Bernoulli beam of rectangular cross-section.
- Application of Hermitian shape functions and Galerkin's method for deriving the elemental mass and stiffness matrices by Finite Element Analysis.
- Free vibration analysis of tapered beam.

2.3 Thesis Organization

The thesis consists of five chapters organized as follows

- Chapter 1 gives an introduction about tapered beam and its application. Literature survey is given
- Chapter 2 is the literature review which gives the idea of the methods used previously for the vibration analysis of tapered beam.

- Chapter 3 provides the formulation of governing differential equation of motion of linearly tapered Euler-Bernoulli beam of rectangular cross-section.
- Chapter 4 provides the Finite Element Formulation of tapered beam of rectangular cross-section.
- Chapter 5 is the result and discussion in which a practical problem is taken which has already been solved by different method and solved by the Finite Element Analysis. Frequencies are calculated for two end conditions i.e. for clamped free and simply supported beam. Effect of taper ratio on the frequency and mode shape is determined.
- Chapter 6 gives the conclusion that we obtained from the theoretical analysis.

THEORETICAL MODELING OF TAPERED BEAM

3.1 Linearly tapered beam element

The beam element is assumed to be associated with two degrees of freedom, one rotation and one translation at each node. The location and positive directions of these displacements in a typical linearly tapered beam element are shown in Fig. 3.1 Some commonly used cross-sectional shapes of beams are shown in Table 3.1. The depth of the cross sections at ends are represented by h_1 (at free end) and h_0 (at fixed end) similarly the width at the both ends are represented by b_1 (at free end) and b_0 (at fixed end) respectively. The length of the element is *l*. The axis about which bending is assumed to take place is indicated by a line in the middle coinciding with the neutral axis.

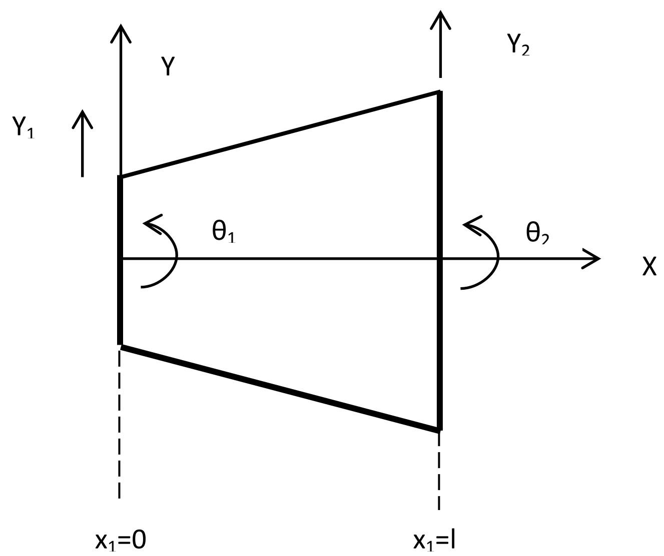


Figure 3.1 the location and positive directions of these displacements in a typical linearly tapered beam element.

For most of the beam shapes the variation in cross-sectional area along the length is represented by the following equation

$$A_x = A_1\left[1+r\frac{x}{l}\right]^m \quad . \tag{3.1}$$

The variation in the moment of inertia along the length about the axis of bending is given as

$$I_x = I_1\left[1+r\frac{x}{l}\right]^n \quad . \tag{3.2}$$

Where

$$r = \frac{h_0}{h_1} - 1 \tag{3.3}$$

A_x and I_x are the cross sectional area and moment of inertia at distance x from the small end; A_1 and I_1 and A_0 and I_0 are the cross sectional area and moment of inertia at free end and fixed end; and m and n refer to the corresponding shape factors that depends on the cross-sectional shape and dimensions of the beam. The shape factors can be evaluated theoretically by observing the Eq. (3.1) and (3.2). Now by applying the condition for the beam as A_x=A_0 and I_x=I_0 at x=l, the condition gives the following equation

$$m = \frac{\ln\left[\frac{A_0}{A_1}\right]}{\ln\left[\frac{h_0}{h_1}\right]}, n = \frac{\ln\left[\frac{I_0}{I_1}\right]}{\ln\left[\frac{h_0}{h_1}\right]} \tag{3.4}$$

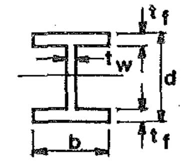
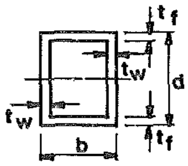
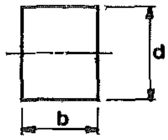
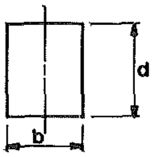
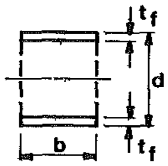
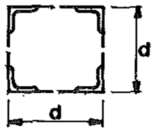
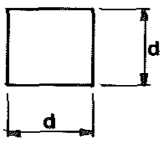
Shape (1)		Shape Factors n (2)	Shape Factors m (3)	Category (4)
	Wide-flange or I-section Constant dimensions b, t_w, t_f Varying depth d Bending about horizontal axis	2.1 to 2.6	varies	1
	Closed box section Constant dimensions b, t_w, t_f Varying depth d Bending about horizontal axis	2.1 to 2.6	varies	
	Solid, rectangular section Constant width b Varying depth d Bending about horizontal axis	3	1	2
	Solid, rectangular section Constant width b Varying depth d Bending about vertical axis	1	1	3
	Open-web section Constant dimensions b, t_f Varying depth d Bending about horizontal axis	2	0	4
	Tower section Constant areas concentrated near corners Varying dimension d	2	0	
	Solid, circular section Varying diameter d	4	2	5
	Solid, square section Varying dimension d	4	2	

Table3.1 Different cross-sectional shapes of tapered beam with shape factors [9]

Thus, the shape factors can be found easily irrespective of the cross-section using the dimensions of at the two ends. Calculation of values of shape factors (*m* and *n)* from Eq. 3. 4 reveals that the expressions for A_x and I_x are exact at both ends of the beam. In some cases as for beams of I-section (Table 3.1), it has been found that, at points in between along the beam, A_x and I_x will deviate slightly from true values. The degree of this deviation is very small and for beams of all usual proportions, Eq. 3.1 and 3.2 gives values of area and moment of inertia at every section along the beam within one percent of deviation, which can be neglected, of the exact values. The shape factors *m* and *n* are dimensionless quantities; *m* and *n* varies between the limits 2.2-2.8.

For theoretical analysis a rectangular cross-sectioned beam with linear variable width and depth is considered.

3.2 Formulation of governing differential equation of tapered beam

A general Euler's Bernoulli beam is considered which is tapered linearly in both horizontal as well as in vertical planes. Fig. 3.2 shows the variation of width and depth in top and front view.

The fundamental beam vibrating equation for Bernoulli-Euler is given by

$$\frac{\partial^2}{\partial x^2}\left(EI\,\frac{\partial^2 y}{\partial x^2}\right)+\frac{\rho A}{g}\left(\frac{\partial^2 y}{\partial t^2}\right)=0 \qquad (3.5)$$

The width and depth are varying linearly given by

$$\begin{aligned} h &= h_1+(h_0-h_1)(x/l) \\ b &= b_1+(b_0-b_1)(x/l) \end{aligned} \qquad (3.6)$$

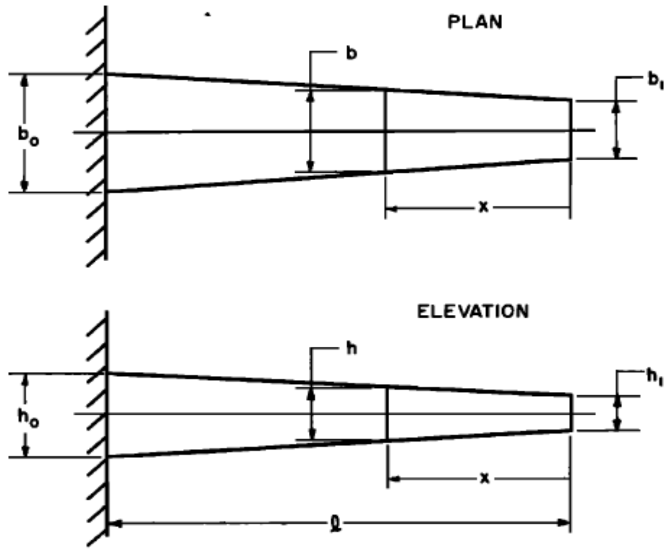


Figure 3.2 Plan and elevation view of cantilever tapered beam with linearly varying width and depth.

Similarly area and moment of inertia will be varying accordingly

$$\begin{aligned} A(x) &= [h_1 + (h_0 - h_1)(x/l)][b_1 + (b_0 - b_1)(x/l)] \\ I(x) &= \frac{1}{12}[b_1 + (b_0 - b_1)(x/l)][h_1 + (h_0 - h_1)(x/l)]^3 \end{aligned} \tag{3.7}$$

All the expressions for the beam area and moment of inertia at any cross-section are written after considering the variation along the length to be linear. Where ρ is the weight density, A is the area, and together 'ρA/g' is the mass per unit length, E is the modulus of elasticity and 'I' is the moment of inertia and l is the length of the beam.

Here we considered only the free vibration, so considering the motion to be of form y(x, t) =z(x) sin (ωt), so applying the following relation to the fundamental beam equation we get

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 z}{\partial x^2}\right) = \frac{\rho A}{g}\left(\omega^2 z\right) \tag{3.8}$$

Substituting Eq. (3.7) and (3.8) into (3.4), and by letting u = x/l (where u varies from 0 to 1) the following equation is obtained

$$\frac{d^4z}{du^4}+\frac{2d^3z}{du^3}\left[\frac{3(h_0-h_1)}{h_1+(h_0-h_1)u}+\frac{(b_0-b_1)}{b_1+(b_0-b_1)u}\right]$$
$$+\frac{6d^2z}{du^2}\left[\begin{array}{l}\dfrac{(b_0-b_1)(h_0-h_1)}{[b_1+(b_0-b_1)u][h_1+(h_0-h_1)u]}\\ +\dfrac{(h_0-h_1)^2}{[h_1+(h_0-h_1)u]^2}\end{array}\right] \tag{3.9}$$
$$=\frac{12l^4\Omega^2}{Eg}\left[\frac{1}{[h_1+(h_0-h_1)u]}\right]^3 z$$

By proper approximation i.e. $\alpha=h_0/h_1$ and $\beta=b_0/b_1$ and $k=12\rho\omega^2/Egh_1^2$ above equation gets transformed into Eq. 11.

$$\frac{d^4z}{du^4}+\frac{2d^3z}{du^3}\left[\frac{3(\alpha-1)}{1+(\alpha-1)u}+\frac{(\beta-1)}{1+(\beta-1)u}\right]$$
$$+\frac{6d^2z}{du^2}\left[\begin{array}{l}\dfrac{(\beta-1)(\alpha-1)}{[1+(\alpha-1)u][1+(\beta-1)u]}\\ +\dfrac{(\alpha-1)^2}{[1+(\alpha-1)u]^2}\end{array}\right] \tag{3.10}$$
$$=\left[\frac{(lk)^2 z}{[1+(\alpha-1)u]^2}\right]$$

The Eq. 3.10 is the final equation of motion for a double-tapered beam with rectangular cross-section. It was solved by numerical integration to give values of (lk) for various taper ratios for clamped-free beam with boundary conditions i.e. at x= 0 or u = 0, d^2z/du^2= 0 and z= 0, at x= l or u= 1, dz/du= 0 and z= 0. This after solving leads to $\omega=k^2h_1\sqrt{\dfrac{Eg}{12\rho}}$. The relation can be used as a comparison while solving with FEA to show the effect of taper ratio on the fundamental frequencies and mode shapes.

FINITE ELEMENT FORMULATION

Most numerical techniques lead to solutions that yield approximate values of unknown quantities i.e. displacements and stiffness, only at selected points in a body. A body can be discretized into an equivalent system of smaller bodies. The assemblage of such bodies represents the whole body. Each subsystem is solved individually and the results so obtained are then contained are then combined to obtain solution for the whole body. Of the numerical techniques, the finite element technique is the most suitable for digital computers. It is applicable to wide range of problems involving non-homogeneous materials, nonlinear stress-strain relations, and complicated boundary conditions. Such problems are usually tackled by one of the three approaches, namely (i) displacement method or stiffness method (ii) the equilibrium or force method and (iii) mixed method. The displacement method, to which we shall confine our discussion, is widely used because of the simplicity with which it can be handled on the computer.

In the displacement approach, a structure is divided into a number of finite elements and the elements are interconnected at joints called as nodes. The displacements in each element are then represented by simple functions. The unknown magnitudes of these functions are the displacements or the derivatives of the displacements at the nodes. A displacement function is generally expressed in terms of polynomial. From the convergence point of view such a function is so chosen that it

i) Is continuous within the elements and compatible between the adjacent elements.

ii) It includes the rigid body displacements and rotations of an element.

iii) Has a consistent strain state.

Further while choosing the polynomial for the displacement function, the order of the polynomial has to be chosen very carefully.

4.1 Calculation of shape function

The analysis of two dimensional beams using finite element formulation is identical to matrix analysis of structures. The Euler-Bernoulli beam equation is based on the assumption that the plane normal to the neutral axis before deformation remains normal to the neutral axis after deformation. Since there are four nodal variables for the beam element, a cubic polynomial function for y(x), is assumed as

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{4.1}$$

From the assumption for the Euler-Bernoulli beam, slope is computed from Eq. (3.1) is

$$\theta(x) = a_1 + 2a_2 x + 3c_3 x^2 \tag{4.2}$$

Where α_0, α_1, α_2, α_3 are the constants The Eq. (3.1) can be written as

$$Y(x) = [1 \quad x \quad x^2 \quad x^3] \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \tag{4.3}$$

$\therefore$ Y(x) = [C] [α]

Where $[C] = [1 \quad x \quad x^2 \quad x^3]$ and $[\alpha] = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$ [4.4]

For convenience local coordinate system is taken $x_1=0$, $x_2=l$ that leads to

$$\begin{aligned} y_1 &= a_0; \\ \theta_1 &= a_1; \\ y_2 &= a_0 + a_1 l + a_2 l^2 + a_3 l^3; \\ \theta_2 &= a_1 l + a_2 l^2 + a_3 l^3; \end{aligned} \tag{4.5}$$

This can be written as

$$\begin{bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \tag{4.6}$$

$\{a\} = [A]\ \{\alpha\}$

$\{\alpha\} = [A]^{-1}\ \{a\}$ [4.7]

Eq. (4.3) can be written as

$Y(x) = [C]\ [A]^{-1}\ \{a\}$ [4.8]

$Y(x) = [H]\ \{a\}$

Where $[H] = [C]\ [A]^{-1}$

$$[A]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-3}{l^2} & \frac{-2}{l} & \frac{3}{l^2} & \frac{-1}{l} \\ \frac{2}{l^3} & \frac{1}{l^2} & \frac{-2}{l^3} & \frac{-1}{l^2} \end{bmatrix} \tag{4.9}$$

$[H] = [H_1(x), H_2(x), H_3(x), H_4(x)]$

Where Hi(x) are called as Hermitian shape function whose values are given below

$$H_0(x) = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}, H_1(x) = x - \frac{2x^2}{l} + \frac{x^3}{l^2}$$
$$H_2(x) = \frac{3x^2}{l^2} - \frac{2x^3}{l^3}, H_3(x) = -\frac{x^2}{l} + \frac{x^3}{l^2}$$

4.2 Stiffness calculation of tapered beam

The Euler-Bernoulli equation for bending of beam is

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) + \rho A\left(\frac{\partial^2 y}{\partial t^2}\right) = q(x,t) \qquad [4.10]$$

Where *y(x, t)* is the transverse displacement of the beam ρ is the mass density, *EI* is the beam rigidity, *q(x, t)* is the external pressure loading, t and x represents the time and spatial axis along the beam axis. We apply one of the methods of the weighted residual, Galerkin's method, to the above beam equation to develop the finite element formulation and the corresponding matrices equations. The average weighted residual of Eq. (4.10) is

$$I = \int_0^l \left(\rho\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y}{\partial x^2}\right) - q\right) p dx = 0 \qquad [4.11]$$

Where l is the length of the beam and *p* is the test function. The weak formulation of the Eq. (4.11) is obtained from integration by parts twice for the second term of the equation. Allowing discretization of the beam into number of finite elements gives

$$I = \sum_{i=1}^{n}\left[\int_{\Psi^e} \rho\frac{\partial^2 y}{\partial x^2} pdx + \int_{\Psi^e} EI(x)\frac{\partial^2 y}{\partial x^2}\frac{\partial^2 p}{\partial x^2} dx - \int_{\Psi^e} qpdx\right] + \left[-Vp - M\frac{dp}{dx}\right]_0^l = 0 \qquad [4.12]$$

Where

$V = -EI(x)\dfrac{\partial^3 y}{\partial x^3}$ is the shear force,

$M = -EI(x)\dfrac{\partial^2 y}{\partial x^2}$ is the bending moment,

Ψ^e is an element domain and n is the number of elements for the beam.

Applying the Hermitian shape function and the Galerkin's method to the second term of the Eq. (4.12) results in stiffness matrix of the tapered beam element with rectangular cross section i.e.

$$[K^e] = \int_0^l [B]^T EI(x)[B]dx \tag{4.13}$$

Where

$$[B] = \{H_0^{"} H_1^{"} H_2^{"} H_3^{"}\} \tag{4.14}$$

and the corresponding element nodal degree of freedom vector

$$\{d^e\} = \{y_1 \theta_1 y_2 \theta_2\}^T \tag{4.15}$$

In Eq. (4.14) double prime denotes the second derivative of the function.

Since the beam is assumed to be homogeneous and isotropic, so, E that is the elasticity modulus can be taken out of the integration and then the Eq. (4.13) becomes

$$K^e = E\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \tag{4.16}$$

Where k_{mn} $(m, n = 1, 4)$ are the coefficients of the element stiffness matrix.

$$k_{mn} = k_{nm} = E\int_0^l I(x) \frac{\partial^2 H_m}{\partial x^2} \frac{\partial^2 H_n}{\partial x^2} dx \tag{4.17}$$

Solving the above equation, we get the respective values of coefficients of the element stiffness matrix for rectangular cross-sectioned beam.

$$[K^e]=E\begin{bmatrix} \frac{6(I_0+I_1)}{l^3} & \frac{2(I_0+2I_1)}{l^2} & -\frac{6(I_0+I_1)}{l^3} & \frac{2(2I_0+I_1)}{l^2} \\ \frac{2(I_0+2I_1)}{l^2} & \frac{I_0+3I_1}{l} & -\frac{2(I_0+2I_1)}{l^2} & \frac{I_0+I_1}{l} \\ -\frac{6(I_0+I_1)}{l^3} & -\frac{2(I_0+2I_1)}{l^2} & \frac{6(I_0+I_1)}{l^3} & \frac{2(2I_0+I_1)}{l^2} \\ \frac{2(2I_0+I_1)}{l^2} & \frac{I_0+I_1}{l} & -\frac{2(2I_0+I_1)}{l^2} & \frac{3I_0+I_1}{l} \end{bmatrix} \quad [4.18]$$

$$\begin{aligned} k_{11} &= \frac{6(I_0+I_1)}{l}, k_{12} = \frac{2(I_0+2I_1)}{l^2} = k_{22} \\ k_{13} &= -\frac{6(I_0+I_1)}{l^3} = k_{31}, k_{14} = \frac{2(2I_0+I_1)}{l^2} = k_{41} \\ k_{22} &= \frac{I_0+3I_1}{l}, k_{23} = -\frac{2(I_0+2I_1)}{l^2} = k_{32} \\ k_{24} &= \frac{I_0+I_1}{l} = k_{42}, k_{33} = \frac{6(I_0+I_1)}{l^3} \\ k_{34} &= \frac{2(2I_0+I_1)}{l^2} = k_{43}, k_{44} = \frac{3I_0+I_1}{l} \end{aligned} \quad [4.19]$$

Here $I_0=b_0d_0^3/12$ and $I_1=b_1d_1^3/12$.

Eq. (4.19) is called as the element stiffness matrix for tapered beam with rectangular cross-sectioned area.

4.3 Mass matrix of tapered beam

Since, for dynamic analysis of beams, inertia force needs to be included. In this case, transverse deflection is a function of *x* and *t*. the deflection is expressed with in a beam element is given below

$$y(x,t)=H_0(x)y_0(t)+H_1(x)\theta_1(t)+H_2(x)y_2(t)+H_3(x)\theta_3(t) \quad [4.20]$$

$$M^e = \rho \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \qquad [4.21]$$

The coefficients of the element stiffness matrix are

$$m_{mn} = m_{nm} = \int_0^l A(x)[H]^T[H]dx \qquad [4.22]$$

$$[M^e] = \rho \begin{bmatrix} \frac{l(10A_1+3A_0)}{35} & \frac{l^2(15A_1+7A_0)}{420} & \frac{9l(A_1+A_0)}{140} & -\frac{l^2(7A_1+6A_0)}{420} \\ \frac{l^2(15A_1+7A_0)}{420} & \frac{l^3(3A_1+5A_0)}{840} & \frac{l^2(6A_0+7A_1)}{420} & -\frac{l^3(A_0+A_1)}{280} \\ \frac{9l(A_0+A_1)}{140} & \frac{l^2(6A_0+7A_1)}{420} & \frac{l(10A_0+3A_1)}{35} & -\frac{l^2(15A_0+7A_1)}{420} \\ -\frac{l^2(6A_0+7A_1)}{420} & -\frac{l^3(A_0+A_1)}{280} & -\frac{l^2(15A_0+7A_1)}{420} & \frac{l^3(5A_0+3A_1)}{840} \end{bmatrix} \qquad [4.23]$$

The above equation is called as consistent mass matrix, where the individual elements of the consistent mass matrices are

$$m_{11} = \frac{l(10A_1+3A_0)}{35}$$

$$m_{12} = \frac{l^2(15A_1+7A_0)}{420} = m_{22}$$

$$m_{13} = \frac{9l(A_1+A_0)}{140} = m_{31}$$

$$m_{14} = -\frac{l^2(7A_1+6A_0)}{420} = m_{41}$$

and

$$
\begin{aligned}
m_{22} &= \frac{l^3(3A_1+5A_0)}{840} \\
m_{23} &= \frac{l^2(6A_0+7A_1)}{420} = m_{32} \\
m_{24} &= -\frac{l^3(A_0+A_1)}{280} = m_{42} \\
m_{33} &= \frac{l(10A_0+3A_1)}{35} \\
m_{34} &= -\frac{l^2(15A_0+7A_1)}{420} = m_{43} \\
m_{44} &= \frac{l^3(5A_0+3A_1)}{840}
\end{aligned}
\qquad [4.24]
$$

The equation of motion for the beam can be written as

$$[M]\{\ddot{\mathrm{d}}\}+[K]\{\mathrm{d}\}=0 \qquad [4.25]$$

We seek to find the natural motion of system, i.e. response without any forcing function. The form of response or solution is assumed as

$$\{\mathrm{d(t)}\}=\{\phi\}\mathrm{e}^{i\omega t} \qquad [4.26]$$

Here $\{\phi\}$ is the mode shape (eigen vector) and ω is the natural frequency of motion. In other words, the motion is assumed to be purely sinusoidal due to zero damping in the system. The general solution turned out to be a linear combination of each mode as

$$\{\mathrm{d(t)}\}=\mathrm{c}_1\{\phi_1\}\mathrm{e}^{i\omega_1 t}+\mathrm{c}_2\{\phi_2\}\mathrm{e}^{i\omega_2 t}+.........+\mathrm{c}_n\{\phi_n\}\mathrm{e}^{i\omega_n t} \qquad [4.27]$$

Here each constant (c_i) is evaluated from initial conditions. Substituting Eq. (4.27) into Eq. (4.25) gives

$$(-\omega^2[\mathrm{M}]+[\mathrm{K}])\{\phi\}\mathrm{e}^{i\omega t}=0 \qquad [4.28]$$

Using the above equation of motion for the free vibration the mode shapes and frequency can be easily calculated.

NUMERICAL ANALYSIS

5.1 Numerical analysis

For numerical analysis a taper beam is considered with the following properties:

Geometrical properties

Width of the beam = b_0 (fixed end) =2.54e-2m, b_1 (free end) =2.54e-2m

Depth of the beam = h_0 (fixed end) = 5.715e-2m, h_1 (free end) =3.51e-2m

$\beta = b_0 / b_1$ and $\alpha = h_0 / h_1$

Length of the beam = 0.762 m

Material properties

Elastic modulus of the beam = 2.109e11 N/m^2

Density = 7995.74 Kg/m^3

Fundamental frequencies for free vibration analysis is calculated by using the equation of motion described in the previous section for two end conditions of beam given below

1) One end fixed and one end free (cantilever beam), which is shown in Fig. 5.1.

2) Simply supported beam, which is shown in Fig. 5.2.

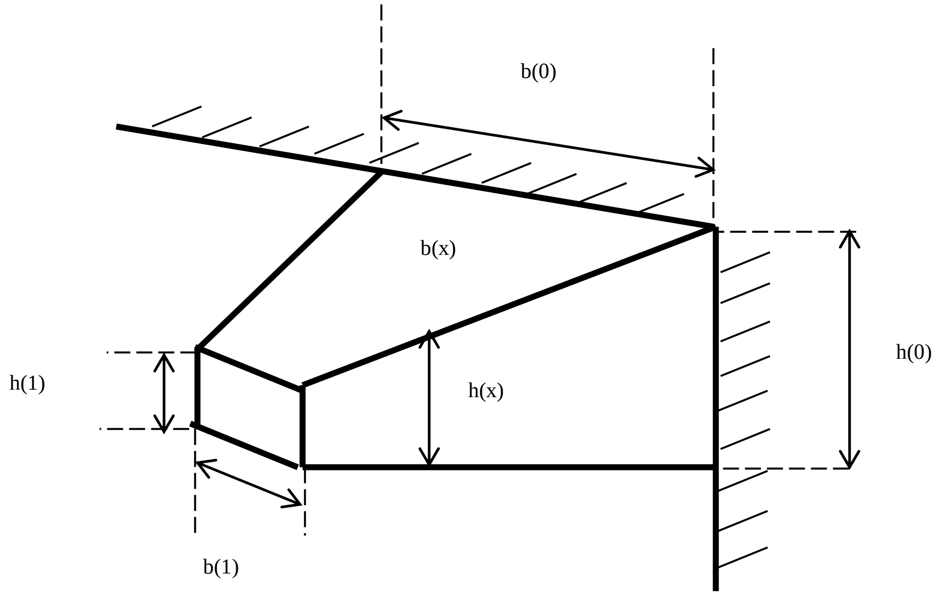


Figure 5.1 Tapered cantilever beam with linearly variable width and depth.

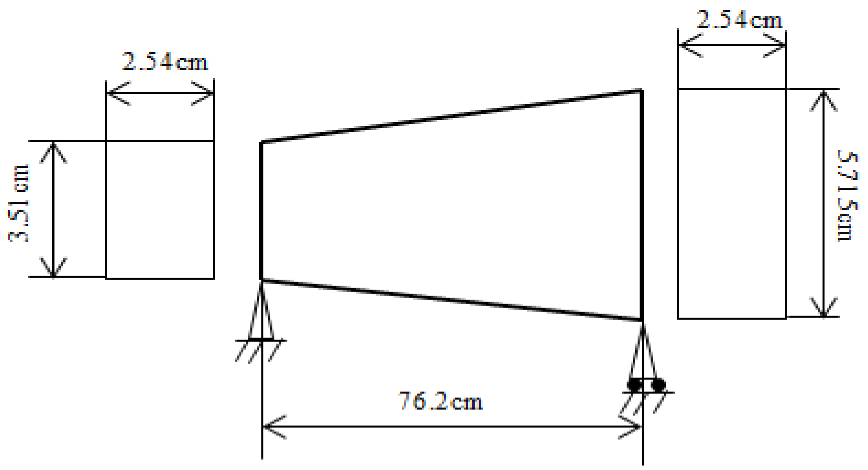


Figure 5.2 Simply supported tapered beam with linearly variable width and depth.

5.2 Calculation of frequency of cantilever beam

After discretizing to 10 numbers of elements, natural frequencies of the tapered beam are calculated using MATLAB program and shown in Table 5.1. As we can see in Fig. 5.3 the frequency converges after discretizing to only four elements. The results are compared with Mabie [1] and Gupta [9] who solved the final differential equation for the beam using numerical method. The method described in here is less cumbersome than the conventional methods.

Table 5.1 Frequencies of linearly tapered cantilever beam.

Frequency (cycles/sec)			Number of Elements							
	1	2	3	4	5	6	7	8	9	10
1	55.8	54.3	53.9	53.1	52.79	52.58	52.46	52.38	52.33	52.29
2	460.48	411.22	403.3	398.29	395.82	394.4	393.61	393.16	392.68	392.4
3		1288.5	1186.8	1178.6	1169.08	1163.8	1160.89	1159	1157.77	1156.8
4		4652.9	2755.6	2642.6	2334.3	2318	2308.2	2302	2298.6	2296.1
5			5198.1	4440.8	3888.7	3879.1	3854.6	3837	3827.1	3820.2
6			11479.2	7147.4	6536.36	5828.4	5813.5	5780	5754.6	5737.1
7				11345.8	9587.54	9037.5	8165.1	8137	8097.4	8060.8
8				20968.4	13910.9	12466.5	11940.2	10902	10851.5	10805
9					19789.1	17045.5	15769.6	15240	14045.3	13955
10					33176	23033	20661	19491	18933	17595
11						30414	26849	24725	23628	23017
12						48142	34445	31172	29222	28178
13							43176	38075	35969	34142
14							65882	56400	44015	41217
15								54112	53412	49527
16								64192	63813	59179
17									69022	70065
18									79985	81661
19										94367
20										135776

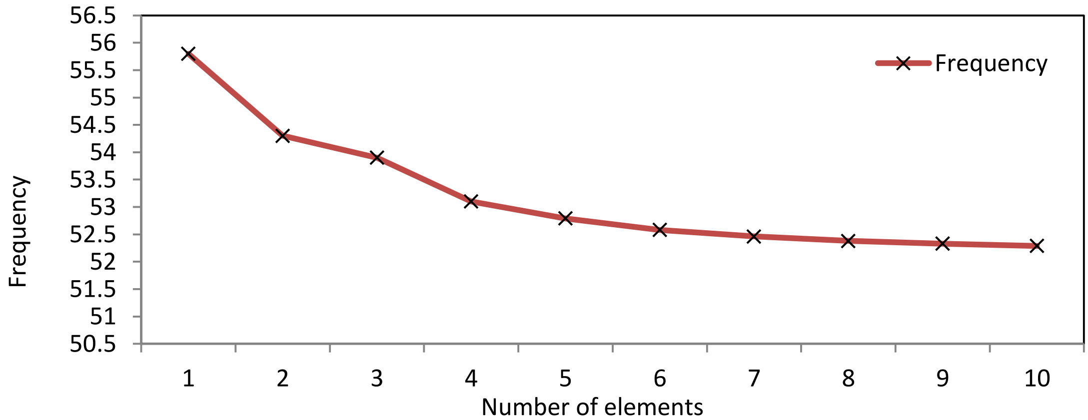


Figure 5.3 Convergence of fundamental natural frequency for cantilever beam

5.3 Calculation of frequency of simply supported beam

The same procedure is applied also for the simply supported beam and natural frequencies are calculated and shown in Table (5.2). The convergence of frequency with number of elements is shown in Fig. 5.4.

Table 5.2 Frequencies of linearly tapered simply supported beam.

Frequency (cycles/sec)	Number of Elements									
	1	2	3	4	5	6	7	8	9	10
1	188.45	187.36	186.34	185.31	184.86	184.45	184.13	184.05	184.02	184.01
2	768.2	762.4	762.4	758.7	757.8	757.7	757.6	757.7	757.7	757.7
3		1913.2	1913.2	1730.1	1714.1	1708.5	1706.1	1705.0	1704.4	1704.2
4		3515.8	3515.8	3388.3	3095.3	3059.6	3045.0	3037.7	3033.8	3031.5
5			6154.0	5352.2	5275.7	4860.7	4797.9	4770.4	4755.3	4746.4
6			9031.7	8428.9	7602.0	7569.9	7028.6	6931.1	6887.1	6861.3
7				12250.2	11044.1	10267.0	10264.6	9601.3	9460.4	9396.6
8				16316.8	15583.4	14078.7	13349.9	13353.2	12580.7	12386.7
9					20269.7	18997.9	17532.3	16853.5	16829.3	15968.8
10					25877.4	24888.5	22798.8	21401.3	20780.6	20686.9
11						30226.9	29213.8	27025.7	25683.3	25132.5
12						37729.7	36214.1	33794.3	31678.0	30377.9
13							42206.7	41639.3	38781.2	36750.8
14							51872.8	49450.4	47060.9	44198.0
15								56311.8	56191.0	52818.1
16								68304.9	64552.6	62564.1
17									72618.5	72769.8
18									87025.8	81554.9
19										91161.1
20										108037.2

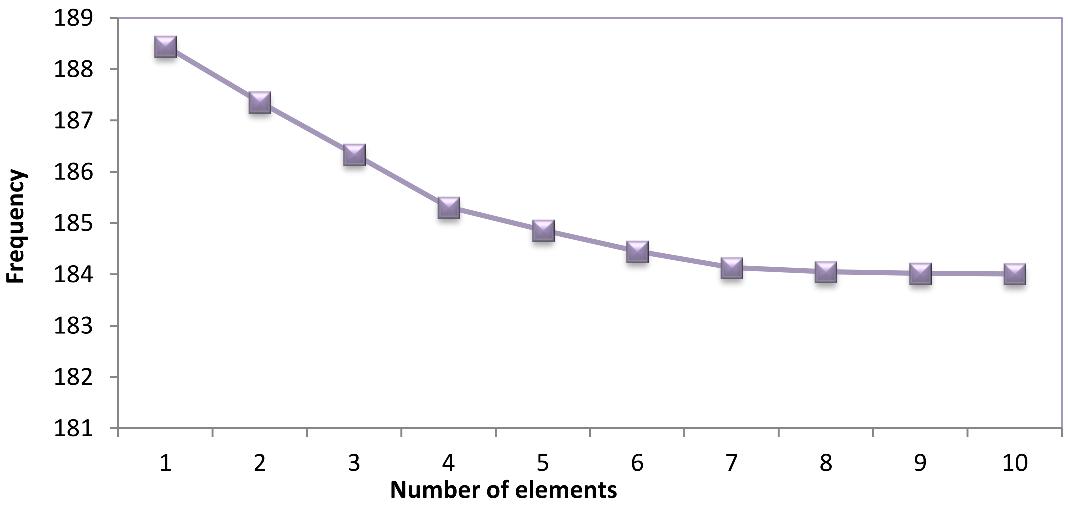


Figure 5.4 Convergence of fundamental natural frequency for simply-supported beam

5.4 Effect of taper ratio on the frequency

For determining the effect of taper ratio on the frequency, different values of α (depth variation) and β (width variation) varying from 1 to 2 are taken in consideration and shown in Table 5.3 to 5.6.

Table 5.3 Frequencies of Linearly Tapered Cantilever Beam for different values of α and β.

FOR β = 1 Natural Frequencies (cycles/sec)					
α	Fundamental Frequency	Second Harmonic	Third Harmonic	Fourth Harmonic	Fifth Harmonic
1	52.05	341.3	958.5	1893.6	3143.02
1.2	52.27	363.76	1044.9	2075.2	3450.9
1.4	52.14	385 32	1128.2	2249.5	3745.3
1.6	53.03	406.16	1209.3	2417.74	4030.45
1.8	53.62	426.43	1288.51	2580.77	4309.72
2.0	54.46	446.22	1366.02	2739.26	4585.43

Table 5.4 Frequencies of Linearly Tapered Cantilever Beam for different values of α and β.

FOR β = 1.2 Natural Frequencies (cycles/sec)					
α	Fundamental Frequency	Second Harmonic	Third Harmonic	Fourth Harmonic	Fifth Harmonic
1	49.13	336.25	955.05	1891.62	3142.24
1.2	49.3	358.31	1041.17	2580.77	3449.05
1.4	49.70	379.46	1124.35	2246.86	3742.42
1.6	50.15	399.91	1205.22	2414.7	4026.8
1.8	50.76	419.78	1284.18	2577.38	4305.75
2.0	51.58	439.18	1361.49	2735.4	4581.30

Table 5.5 Frequencies of Linearly Tapered Cantilever Beam for different values of α and β.

FOR β = 1.4 Natural Frequencies (cycles/sec)					
α	Fundamental Frequency	Second Harmonic	Third Harmonic	Fourth Harmonic	Fifth Harmonic
1	46.77	332.03	952.09	1889.86	3141.23
1.2	47.02	353.75	1038.07	2070.9	3447.09
1.4	47.37	374.56	1121.11	2244.60	3739.71
1.6	47.83	394.67	1201.83	2412.16	4023.68
1.8	48.45	414.21	1280.63	2574.45	4302.29
2.0	49.29	433.27	1357.78	2732.16	4577.80

Table 5.6 Frequencies of Linearly Tapered Cantilever Beam for different values of α and β.

FOR β = 1.6 Natural Frequencies (cycles/sec)					
α	Fundamental Frequency	Second Harmonic	Third Harmonic	Fourth Harmonic	Fifth Harmonic
1	44.83	328.43	949.56	1888.27	3140.08
1.2	45.09	349.84	1035.41	2069.12	3445.16
1.4	45.44	370.37	1118.34	2242.58	3737.18
1.6	45.91	390.19	1198.94	2409.86	4020.79
1.8	46.54	409.44	1277.61	2571.83	4299.26
2.0	47.39	428.2	1354.63	2729.21	4574.82

One can see from Table 5.3 to 5.6 as α value increases there is an increase in the values of fundamental frequency, but the taper ratio β in the horizontal plane has a detrimental effect on frequency. This is a very useful concept that can be used in structures or machine members where strength to weight ratio is important to be considered for minimal weight and highest strength, simultaneously increasing the fundamental frequency.

5.5 Mode shapes

To determine the effect of β and α on the amplitude of vibration, their values are varied (1 to 2) and mode shapes are calculated and shown in Fig. 5.5 to 5.12

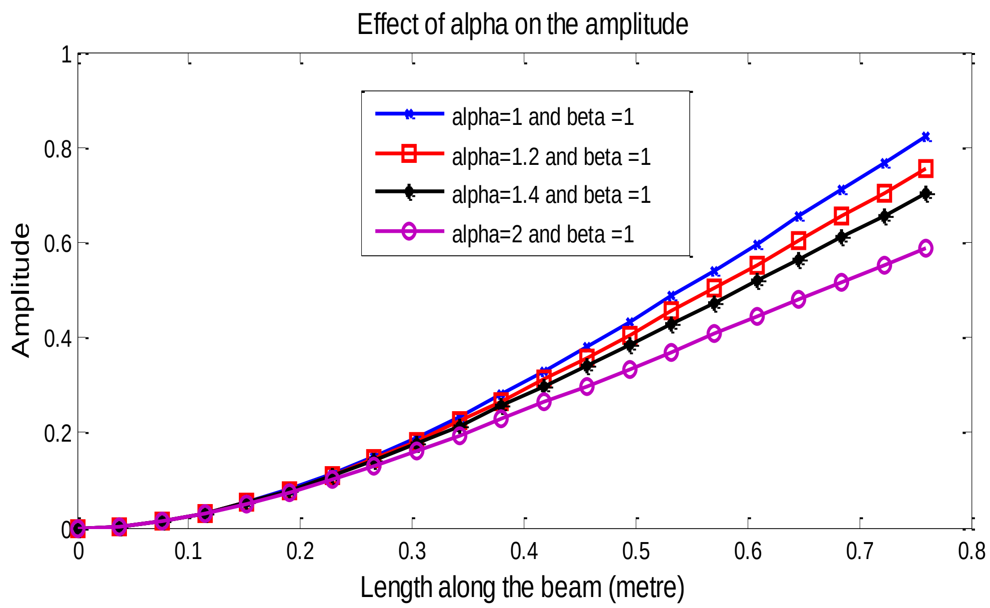


Figure 5.5 First mode shape for cantilever beam constant width (β) and variable depth ratio (α).

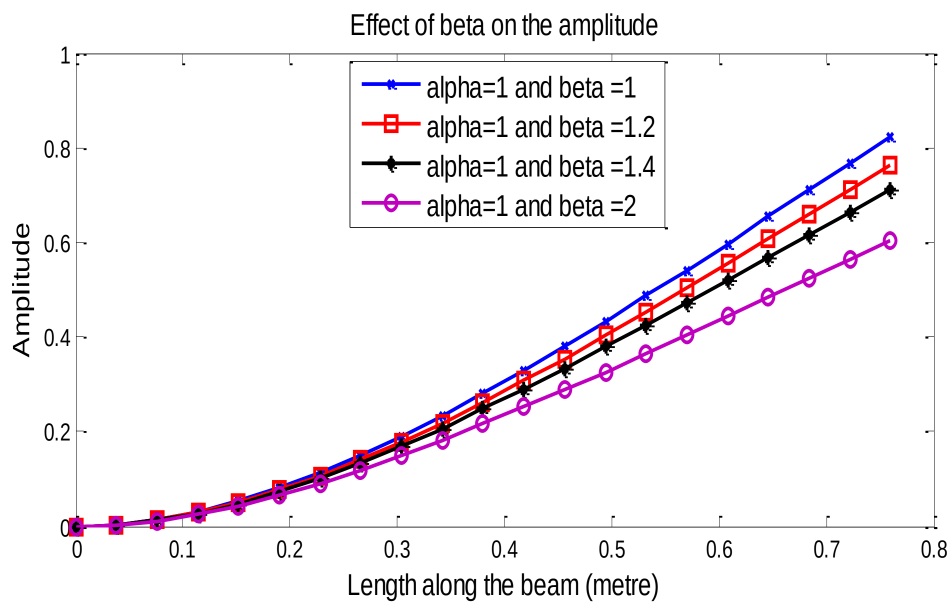


Figure 5.6 First mode shape for cantilever beam constant depth ratio (α) and variable width (β).

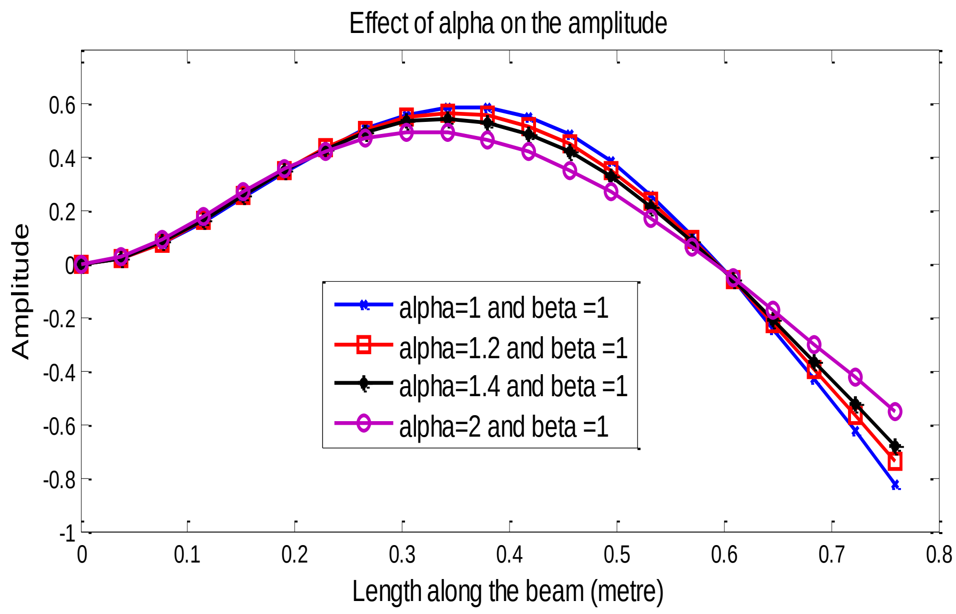


Figure 5.7 Second mode shape for cantilever beam with constant width (β) and variable depth ratio (α).

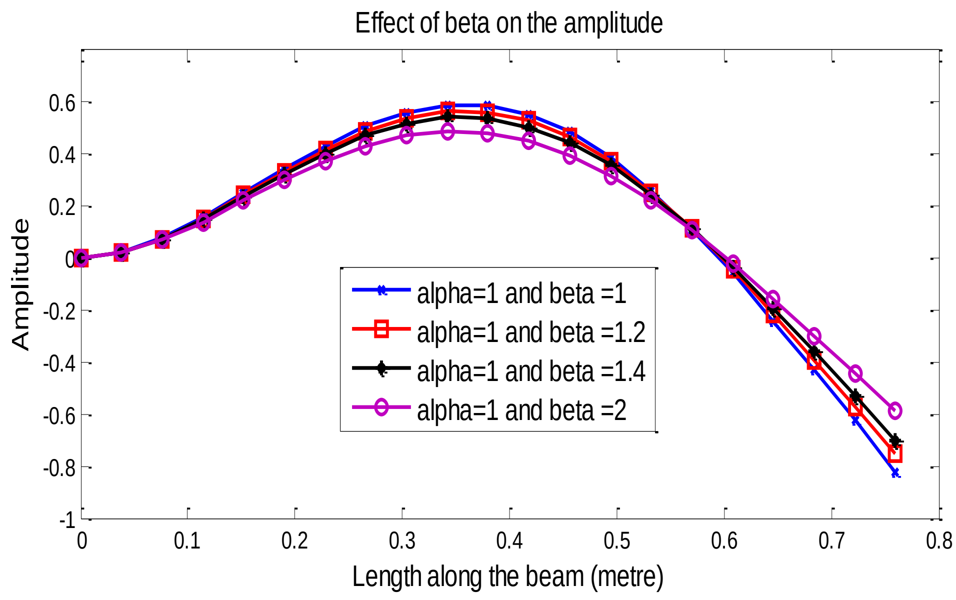


Figure 5.8 Second mode shape for cantilever beam with constant depth (α) and variable width (β) ratio.

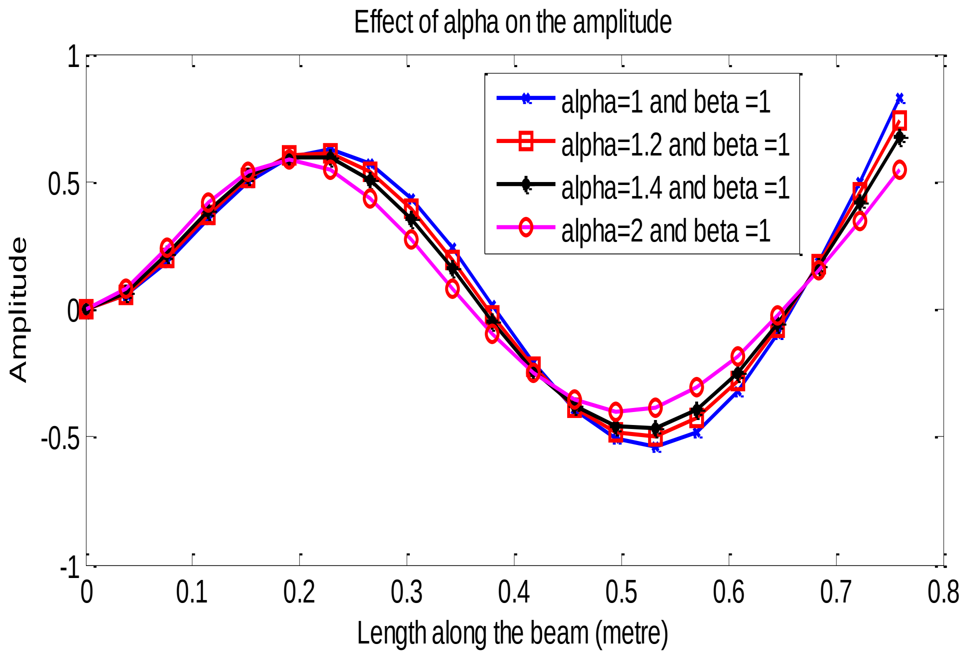


Figure 5.9 Third mode shape for cantilever beam with constant width (β) and variable depth ratio (α).

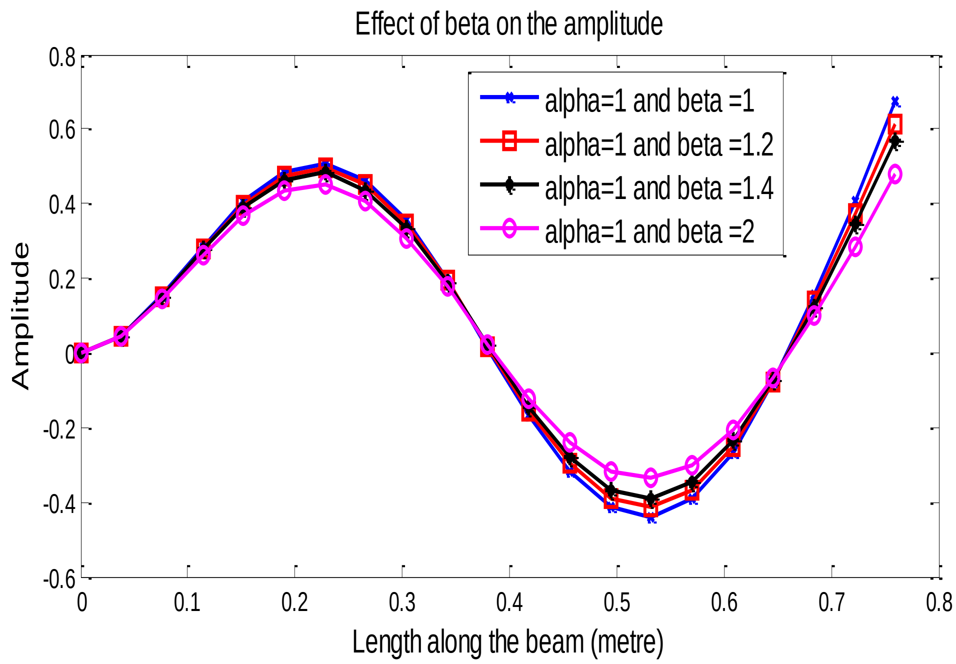


Figure 5.10 Third mode shape for cantilever beam with constant depth (α) and variable width (β) ratio.

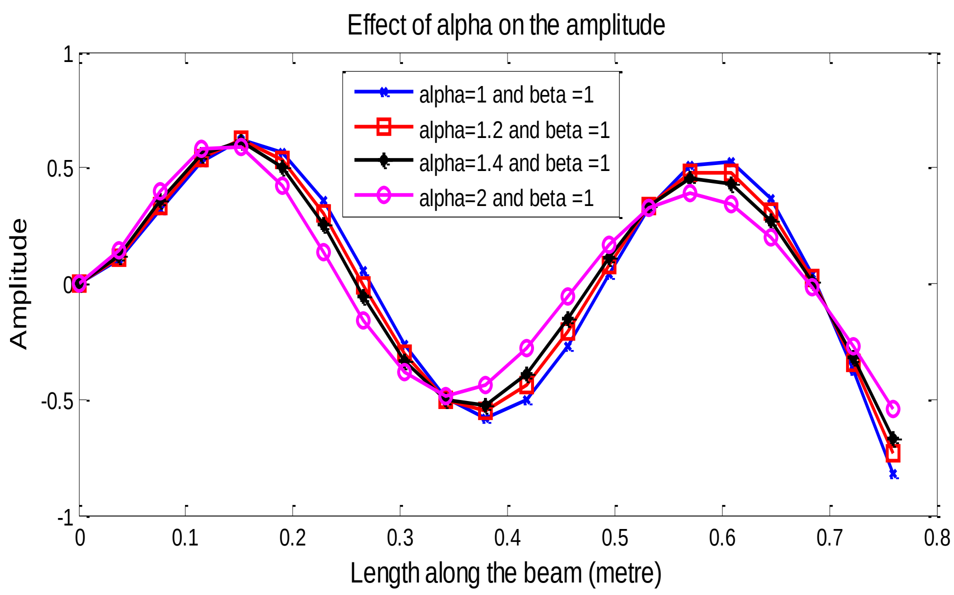


Figure 5.11 Fourth mode shape for cantilever beam constant width (β) and variable depth ratio (α).

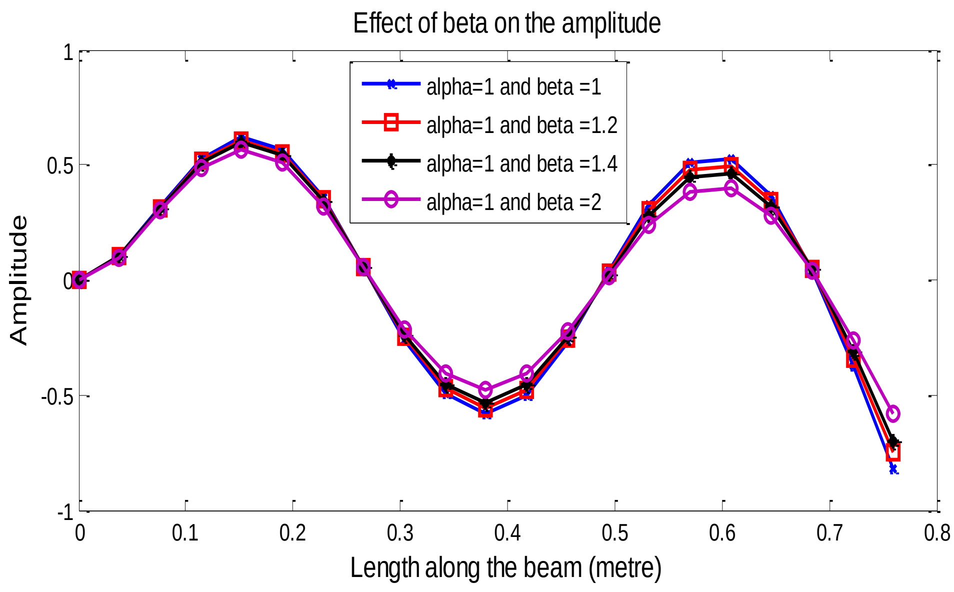


Figure 5.12 Fourth mode shape for cantilever beam constant depth ratio (α) and variable width (β).

The effects of taper ratio (α and β) on the first, second, third and fourth mode shapes are nearly the same, that is of decrease in amplitude of vibration with an increment in their values, but the amount by which the amplitude value decreases is more in higher mode of vibration than in lower mode. This result is shown in Table 5.7 to 5.10.

Table 5.7 Effect of variation of taper ratio on amplitude for First mode.

Taper ratio	% age reduction in amplitude for First mode	
	β	α
1.2	7.54	8.3
1.4	13.63	18.2
2	26.62	28.34

Table 5.8 Effect of variation of taper ratio on amplitude for Second mode.

Taper ratio	% age reduction in amplitude for Second mode	
	β	α
1.2	8.34	9.98
1.4	14.88	17.63
2	28.41	32.88

Table 5.9 Effect of variation of taper ratio on amplitude for Third mode.

Taper ratio	% age reduction in amplitude for Third mode	
	β	α
1.2	8.43	10.32
1.4	15.11	18.28
2	28.82	34.03

Table 5.10 Effect of variation of taper ratio on amplitude for Fourth mode.

Taper ratio	% age reduction in amplitude for Fourth mode	
	β	α
1.2	8.46	10.42
1.4	15.19	18.44
2	28.93	34.25

Maximum values of amplitude is taken for each mode and for different values of taper ratios the % age reduction in amplitude with respect to no taper is calculated and shown in the Table 5.7 to 5.10. As one can see from Table 5.7 to 5.10, the effect of α on decreasing the amplitude is quite more than β.

CONCLUSIONS AND FUTURE WORK

6.1 Conclusions

This research work presents a simple procedure to obtain the stiffness and mass matrices of tapered Euler's Bernoulli beam of rectangular cross section. The proposed procedure is verified by the previously produced results and method. The shape functions, mass and stiffness matrices are calculated for the beam using the Finite Element Method, which requires less computational effort due to availability of computer program. The value of the natural frequency converges after dividing into smaller number of elements. It has been observed that by increasing the taper ratio 'α' the fundamental frequency increases but an increment in taper ratio'β' leads to decrement in value of fundamental frequency. Mode shapes for different taper ratio has been plotted. Taper ratio'α' is more effective in decreasing the amplitude in vertical plane, than β in the horizontal plane at higher mode. The above result is of great use for structural members where high strength to weight ratio is required.

6.2 Future work

Future work in this direction will be aimed at applying the above method and results for the tapered beam and choreographing it with twisted beam which has great application worldwide as if in turbine blades, helicopter blades etc. Instead of considering Euler-Bernoulli beam, Timoshenko beam can be taken which gives more practical idea about the vibrational frequency and mode shapes. End mass can also be incorporated to the tapered beam and modal analysis can be done.

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[28] P. Henri Gavin, "Structural element stiffness matrices and mass matrices," Department of Civil and Environmental Engineering CEE 541 Structural Dynamics, 2012. Fig. 2 Cantilever Beam of rectangular cross-section area with width and depth varying linearly.